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Approximation with Kernels of Finite Oscillations
II. Degree of Approximation

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1. INTRODUCTION

Let C_a denote the class of continuous and $2a$-periodic (or bounded if $a = \infty$) functions defined on the real line, and let $\mathscr{L}(f; x)$ denote the transform of $f \in C_a$ by the linear operator $\mathscr{L}: C_a \to C_a$.

In this paper we investigate the degree of approximation to functions $f \in C_a$ by sequences of certain types of *nonpositive* linear operators in the Tchebycheff norm. The motivation for this investigation is the fact that certain types of sequences of positive linear operators, $\{\mathscr{L}_n\}$, can only achieve a degree of approximation as good as n^{-2} for any interesting class of functions. This has been established by Korovkin [5] for sequences of positive linear trigonometric polynomial operators of degree n, and by Butzer [1] for positive linear operators of the type

$$\mathscr{H}_n(f; x) = n \int_{-\infty}^{\infty} f(x + u)\, H(nu)\, du, \qquad f \in C_\infty, \tag{1.1}$$

where $H(t)$ is a positive, even function continuous at 0, with $\int_{-\infty}^{\infty} H(t)\, dt = 1$. In order to improve this degree of approximation, we introduced in [4], a special type of nonpositive operator, namely, a *2k-zero operator* of the form

$$\mathscr{L}^{(k)}(f; x) = \int_{-a}^{a} f(t)\, K^{(k)}(t - x)\, dt, \qquad 0 < a \leqslant \infty, \qquad f \in C_a,$$

where $K^{(k)}(u) \in C_a$ is an even function that oscillates across the u-axis a fixed number of times in a prescribed manner (see Definitions 1 and 2). Necessary and sufficient conditions were given in [4] for sequences, $\{\mathscr{L}_n^{(k)}(f; x)\}$, of such $2k$-zero approximating functions to converge uniformly to $f(x) \in C_a$.

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In the present paper, it is shown that for sequences of $2k$-zero trigonometric polynomial operators, $\{\mathscr{T}_n^{(k)}\}$, and $2k$-zero operators, $\{\mathscr{H}_n^{(k)}\}$, of the type (1.1) (see Definition 3), better degrees of approximation are possible for certain classes of functions. In particular, 2-zero operators may yield better degrees of approximation than is possible with positive linear operators of these types. On the other hand, it is shown that no matter how "smooth" $f \in C_a$ is, the degree of approximation to f is not better than n^{-2k-2} for these types of $2k$-zero operators. These results are contained in Theorems 2, 3a and 3b. Further, a method for constructing $2k$-zero operators is discussed (Theorem 1) and some examples are given.

Definitions

Let $\mu(t)$ be an analytic, even function defined on $[-a, a]$ such that $\mu(0) = 0$ and $\mu(t)$ is strictly increasing on $[0, a]$. Denote the jth *μ-moment of* $f \in C_a$ by $M_j(\mu, f)$, i.e.,

$$M_j(\mu, f) = \int_{-a}^{a} \mu^j(t) f(t)\, dt, \qquad j = 0, 1, \ldots,$$

and set $M_0(f) = M_0(\mu, f)$ whenever convenient.

We call α a *simple zero* of a function $f \in C_a$ if $f(\alpha) = 0$, and for some $\epsilon > 0$, $\alpha - \epsilon < \zeta_1 < \alpha < \zeta_2 < \alpha + \epsilon$ implies $f(\zeta_1) f(\zeta_2) \neq 0$, $\operatorname{sgn}[f(\zeta_1)] = -\operatorname{sgn}[f(\zeta_2)]$.

Definition 1. A function $K(t) \in C_a$ is called a *kernel* if

(i) $K(t) = K(-t)$,
(ii) $M_0(K) = 1$.

If, in addition, $K(t)$ has exactly k simple zeros α_i, $i = 1, 2, \ldots, k$, in $(0, a)$ and for some $\mu(t)$ we have

(iii) $M_j(\mu, K) = 0, j = 1, \ldots, k$,

then $K(t)$ is called a *$2k$-zero kernel* with respect to μ, and is denoted by $K^{(k)}(t)$.

Definition 2. Let $\mathscr{L}^{(k)}\colon C_a \to C_a$ be the linear operator defined by the convolution

$$\mathscr{L}^{(k)}(f; x) = \int_{-a}^{a} f(t)\, K^{(k)}(t - x)\, dt, \qquad 0 < a \leqslant \infty, \qquad f \in C_a,$$

where $K^{(k)}$ is a $2k$-zero kernel. Then $\mathscr{L}^{(k)}$ is called a *$2k$-zero operator*.

DEFINITION 3. (a) Let $a = \pi$ and $\mu(t) = \sin^2(t/2)$. For a fixed k and each n, let $\mathscr{L}_n^{(k)}$ be a $2k$-zero operator such that $\mathscr{L}_n^{(k)}(f; x)$ is a trigonometric polynomial of degree n. Denote these operators by $\mathscr{T}_n^{(k)}$ and set $M_j^T(f) = M_j[\sin^2(t/2), f]$, $j \geqslant 0$. (Note that $\mathscr{T}_n^{(k)}(f; x)$ is a trigonometric polynomial of degree n, for every $f \in C_a$, if and only if $K_n^{(k)}(t)$ is a trigonometric polynomial of degree n.)

(b) Let $a = \infty$ and $\mu(t) = t^2$. For a given k and a given $2k$-zero kernel $H^{(k)}$ with respect to μ, define $K_n^{(k)}(t) = nH^{(k)}(nt)$, $n = 1, 2,\ldots$. Denote the operator defined by the $2k$-zero kernel $K_n^{(k)}$ by $\mathscr{H}_n^{(k)}$ and set $M_j^H(f) = M_j(t^2, f)$, $j \geqslant 0$. It is assumed in this case that $M_{k+1}^H(| H^{(k)} |)$ exists.

2. CONSTRUCTION OF $2k$-ZERO KERNELS

Method

The construction consists of multiplying a positive, even function $K(t) \in C_a$ by an appropriate even function $p_\lambda(t) \in C_a$ which has exactly $2k$ zeros in $(-a, a)$.

We show that if $\lambda(t)$ is a function defined on $[-a, a]$, satisfying certain conditions, then there exists an algebraic polynomial $p(x)$ of degree k such that we can take $p_\lambda(t) = p(\lambda(t))$, $t \in [-a, a]$. Furthermore, we have then $M_0(\lambda, p_\lambda k) = 1$ and $M_j(\lambda, p_\lambda K) = 0$, $j = 1,\ldots, k$. We first note the following fact:

Fact 1. Let $f(t)$ ($\not\equiv 0$) be a nonnegative, even, continuous function on $[-a, a]$. Let $\lambda(t)$ be defined on $[-a, a]$ and suppose $\{\lambda^i(t)\}_{i=0}^k$ is linearly independent on $[-a, a]$. Then, if the λ-moments $M_j(\lambda, f)$, $j = 0,\ldots, 2k$, exist, the system of equations

$$\sum_{i=0}^{k} \beta_i M_{i+j}(\lambda, f) = \delta_j\,, \qquad j = 0, 1,\ldots, k, \tag{2.2}$$

where $\delta_0 = 1$ and $\delta_j = 0$, $j \geqslant 1$, has a unique solution.

The proof of this fact is accomplished by showing that the coefficient matrix of (2.2) represents a positive definite quadratic form (Hoff [3]).

THEOREM 1. *Let $K(t) \in C_a$ be a nonnegative ($\not\equiv 0$) even function and let $\mu(t)$ be as in Section 1. If the μ-moments $M_j(\mu, K)$, $j = 0, 1,\ldots, 2k$, exist, then there is a polynomial $p(x) = \sum_{i=0}^k \gamma_i x^i$ such that if we set $p_\mu(t) = p(\mu(t))$ for $| t | \leqslant a$ and extend $p_\mu(t)$ periodically to the whole real line if $a < \infty$, then $p_\mu(t) K(t)$ is a $2k$-zero kernel.*

Proof. Since $\mu(t)$ is strictly increasing on $[0, a]$, $\{\mu^i(t)\}_{i=0}^k$ is linearly independent on $[-a, a]$. By Fact 1 with $\lambda = \mu$ and $f = K$, there exists a solution $\{\beta_i\}_{i=0}^k$ to the system (2.2). Let $q(x) = \sum_{i=0}^k \beta_i x^i$ and set $q_\mu(t) = q(\mu(t))$, $t \in [-a, a]$. If $a < \infty$, extend $q_\mu(t)$ to the whole real line as a $2a$-periodic function. Then $q_\mu(t) \in C_a$ is even and $q_\mu(t)\, K(t)$ satisfies conditions (i), (ii), and (iii) of the definition of a $2k$-zero kernel with respect to μ (Definition 1). Furthermore, since $q_\mu(t)\, K(t)$ satisfies condition (iii) of Definition 1, $q_\mu(t)$ must have at least $2k$ simple zeros in $(-a, a)$ (Hoff [3]). On the other hand, $q(x)$ has at most k simple zeros in $(-a, a)$. Therefore, since $\mu(t)$ is even and strictly monotone on $[0, a]$, $q_\mu(t) = q(\mu(t))$ has at most $2k$ simple zeros in $(-a, a)$. Thus, with $p(x) = q(x)$, the theorem is established.

Examples

We first construct 2-zero and 4-zero kernels of the form $nH(nt)$ on $(-\infty, \infty)$. Consider the positive Weierstrass kernels

$$W_n^{(0)}(t) = \frac{n}{\sqrt{\pi}} e^{-(nt)^2}$$

and note that $M_j^H(W_n^{(0)}) = (1 \cdot 3 \cdots (2j-1))/(2n^2)^j$, $j \geq 1$, and $M_0^H(W_n^{(0)}) = 1$. Solve (2.2), with $f = W_n^{(0)}$, $\lambda(t) = t^2$ and $k = 1$. This yields the solution $\beta_0 = \frac{3}{2}$ and $\beta_1 = -n^2$. Therefore, we have the sequence of 2-zero kernels

$$W_n^{(1)}(t) = \frac{n}{\sqrt{\pi}} \left(\frac{3}{2} - (nt)^2\right) e^{-(nt)^2}$$

with zeros $\pm\alpha_{n1} = \pm(1/n)\sqrt{\frac{3}{2}}$. Similarly, with $k = 2$, we construct the 4-zero kernels

$$W_n^{(2)}(t) = \frac{n}{\sqrt{\pi}} \left(\frac{15}{8} - \frac{5}{2}(nt)^2 + \frac{1}{2}(nt)^4\right) e^{-(nt)^2}$$

with zeros

$$\pm\alpha_{n1} = \pm\frac{1}{n}\sqrt{\frac{5}{2} - \sqrt{\frac{5}{2}}} \qquad \text{and} \qquad \pm\alpha_{n2} = \pm\frac{1}{n}\sqrt{\frac{5}{2} + \sqrt{\frac{5}{2}}}.$$

As another example, consider the positive nth degree trigonometric polynomial

$$V_n^{(0)}(t) = \left(\cos\frac{t}{2}\right)^{2n},$$

and note that, with $\rho = \int_{-\pi}^{\pi} (\cos(t/2))^{2n}\, dt$, we have

$$M_j^T(V_n^{(0)}) = \rho \frac{1 \cdot 3 \cdots (2j-1)}{2^j(n+j)(n+j-1)\cdots(n+1)} = O(n^{-j}).$$

Solve (2.2), with $f = (1/\rho)\, V_n^{(0)}$, $\lambda(t) = (\sin(t/2))^2$ and $k = 1$. This yields the sequence of 2-zero trigonometric polynomial kernels of degree $n + 1$:

$$V_{n+1}^{(1)}(t) = \frac{n+1}{\rho(2n+1)} \left(3 - (2n+4)\left(\sin\frac{t}{2}\right)^2\right)\left(\cos\frac{t}{2}\right)^{2n}.$$

Similarly, with $k = 2$, we have the 4-zero kernels of degree $n + 2$:

$$V_{n+2}^{(2)}(t) = \frac{(n+1)(n+2)}{2\rho(4n^2+8n+3)} \left(5 - 20(n+3)\left(\sin\frac{t}{2}\right)^2 + 4(n+3)(n+4)\left(\sin\frac{t}{2}\right)^4\right)\left(\cos\frac{t}{2}\right)^{2n}.$$

The zeros of these 2-zero and 4-zero trigonometric kernels are all of the order $O(n^{-1/2})$.

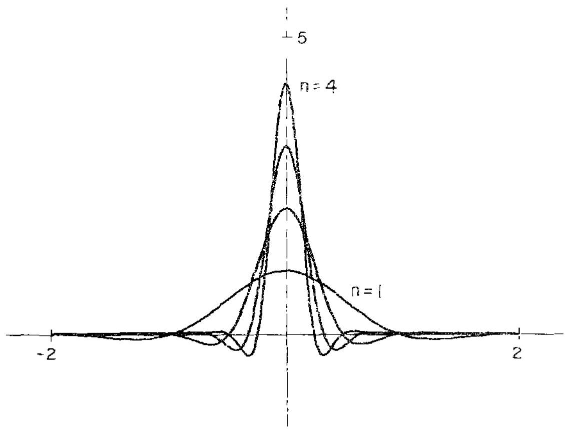

FIG. 1. Kernels $W_n^{(2)}$, $n = 1, 2, 3, 4$.

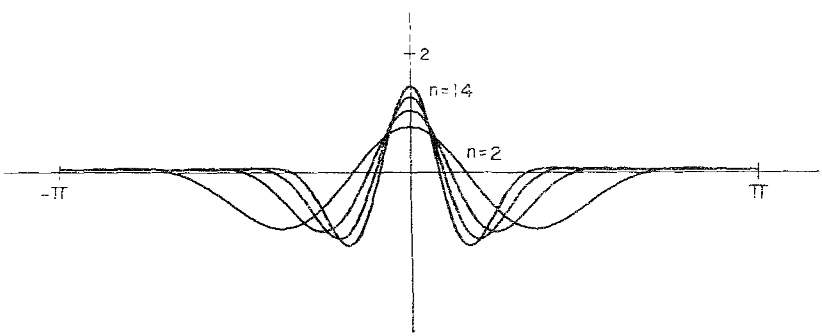

FIG. 2. Kernels $V_{n+2}^{(2)}$, $n = 2, 6, 10, 14$.

We note further that $\{M_0{}^H(|\, W_n^{(j)}\,|)\}$ and $\{M_0{}^T(|\, V_n^{(j)}\,|\}$, $j = 1, 2$, are bounded sequences. Therefore, from the results in Part I on convergence, since the sequences of kernels $\{W_n^{(j)}\}$ and $\{V_n^{(j)}\}$, $j = 1, 2$, peak, each one defines a sequence of 2-zero or 4-zero operators which, when applied to an $f \in C_a$, converge uniformly to it.

A Special Example

We now construct a special sequence of $2k$-zero trigonometric polynomial kernels. These kernels are used later in proving Theorem 3b concerning the degree of approximation of trigonometric operators. The kernels are constructed from generalized Jackson kernels, i.e., positive trigonometric polynomials of degree $2(n-1)(k+1)$ of the form

$$J_{nk}(t) = n^{-4k-3}\left(\frac{\sin\dfrac{nt}{2}}{\sin\dfrac{t}{2}}\right)^{4k+4}, \qquad k = 1, 2,\ldots. \tag{2.3}$$

We first state, without proof, the following two facts (Hoff [3]):

Fact 2. For $k \geqslant 0$, let

$$s_{jn} = n^{-4k-3}\int_{-\pi}^{\pi}\frac{\left(\sin\dfrac{nt}{2}\right)^{4k+4}}{\left(\sin\dfrac{t}{2}\right)^{4k+4-2j}}\,dt, \tag{2.4}$$

$$\sigma_j = 4\int_0^\infty \frac{(\sin t)^{4k+4}}{t^{4k+4-2j}}\,dt < \infty, \tag{2.5}$$

and

$$\hat{s}_{jn} = \sigma_j n^{-2j}, \qquad j = 0, 1,\ldots, 2k+1. \tag{2.6}$$

Then

$$s_{jn} = \hat{s}_{jn} + o(n^{-2j}).$$

Fact 3. Let $\hat{s}_{jn}$ be defined by (2.6). Let $\hat{S}$ be the $(k+1)\times(k+1)$ matrix

$$\hat{S} = \begin{bmatrix} \hat{s}_{0n} & \hat{s}_{1n} & \cdots & \hat{s}_{kn} \\ \hat{s}_{1n} & \hat{s}_{2n} & \cdots & \vdots \\ \vdots & & & \vdots \\ \hat{s}_{kn} & & \cdots & \hat{s}_{2k,n} \end{bmatrix}$$

and let $\hat{S}_i$ be the submatrix of $\hat{S}$ obtained by crossing out the first row and ith column of $\hat{S}$. Then there are constants $b_0 \neq 0$, $b_1,\ldots, b_{k+1}$ such that with $k' = 2k(k+1)$, we have $\det\hat{S} = b_0 n^{-k'}$ and $\det\hat{S}_i = b_i n^{-k'+2(i-1)}$, $i = 1,\ldots, k+1$. Furthermore, if $\rho(m)$ is a permutation of the integers $m = 0, 1,\ldots, k$, and if we set $m_\rho = m + \rho(m)$, then

$$\prod_{m=0}^{k} \hat{s}_{m_\rho n} = O(n^{-k'}).$$

LEMMA 1. *Let J_{nk} be defined by* (2.3), $k \geqslant 1$. *Then there exist $2k$-zero kernels of the form*

$$\hat{K}_N^{(k)}(t) = \left[\sum_{i=0}^{k} c_{in}^{(k)} \left(\sin \frac{t}{2}\right)^{2i}\right] J_{nk}(t), \tag{2.7}$$

where $c_{in}^{(k)} = O(n^{2i})$ as $n \to \infty$, $i = 0, 1,..., k$ and $N = 2(n-1)(k+1)$.

Proof. Let s_{jn} be defined by (2.4) and consider the system of equations

$$\sum_{j=0}^{k} s_{j+i,n} x_i = \delta_i, \qquad i = 0, 1,..., k, \tag{2.8}$$

where $\delta_0 = 1$, $\delta_i = 0$, $i \geqslant 1$. Since $M_j{}^T(J_{nk}) = s_{jn}$, therefore, by Fact 1, with $\lambda(t) = \sin^2(t/2)$ and $f = J_{nk}$, a solution $\{x_i\} = \{c_{in}^{(k)}\}$, $i = 0,..., k$, exists for (2.8). Hence, by Theorem 1, with $q(x) = \sum_{i=0}^{k} c_{in}^{(k)} x^i$ and $\mu(t) = \sin^2(t/2)$, the functions (2.7) are $2k$-zero trigonometric polynomial kernels.

We now show $c_{in}^{(k)} = O(n^{2i})$, $i = 0,..., k$. Note, first, that by Fact 2, $s_{jn} = \hat{s}_{jn} + o(n^{-2j})$, where $\hat{s}_{jn}$ is defined by (2.6). This may be rewritten in the form $s_{jn} = \hat{s}_{jn}(1 + \epsilon_{jn})$, where $\epsilon_{jn} \to 0$ as $n \to \infty$.

Let S be the coefficient matrix of the system of equations (2.8):

$$S = \begin{bmatrix} s_{0n} & s_{1n} & \cdots & s_{kn} \\ s_{1n} & s_{2n} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ s_{kn} & & \cdots & s_{2k,n} \end{bmatrix},$$

and let S_i denote the submatrix of S obtained by crossing out the first row and ith column of S. We show that, with $k' = 2k(k+1)$, $\det S = b_0 n^{-k'} + o(n^{-k'})$ and $\det S_i = O(n^{-k'+2(i+1)})$, $i = 1,..., k+1$, where b_0 is some nonzero constant. Let $\rho(m)$ denote a permutation of the integers $m = 0, 1,..., k$, and let $m_\rho = m + \rho(m)$. Then

$$\begin{aligned} \det S &= \sum_{\rho} \left(\operatorname{sgn} \rho \prod_{m=0}^{k} s_{m_\rho n}\right) \\ &= \sum_{\rho} \left(\operatorname{sgn} \rho \prod_{m=0}^{k} \hat{s}_{m_\rho n}(1 + \epsilon_{m_\rho n})\right) \\ &= \sum_{\rho} \left(\operatorname{sgn} \rho \prod_{m=0}^{k} \hat{s}_{m_\rho n}\right) + D_n \\ &= \det \hat{S} + D_n, \end{aligned} \tag{2.9}$$

where $\hat{S}$ is the matrix defined in Fact 3, and D_n is a sum of terms of the form

$$\left(\prod_{m=0}^{k} \hat{S}_{m_p n}\right)\left(\prod_{m=0}^{l} \eta_{mn}\right), \qquad l \leqslant k,$$

where each η_{mn} is one of the $\epsilon_{m_p n}$. By Fact 2, $\prod_{m=0}^{k} s_{m_p n} = O(n^{-k'})$. Hence, since $\eta_{mn} \to 0$ as $n \to \infty$, $D_n = o(n^{-k'})$. Also, by Fact 2, we have $\det \hat{S} = b_0 n^{-k'}$, $b_0 \neq 0$. Therefore, from (2.9) we obtain $\det S = b_0 n^{-k'} + o(n^{-k'})$, $b_0 \neq 0$. In a similar manner, it can be shown that $\det S_i = O(n^{-k'+2(i-1)})$, $i = 1,..., k+1$.

Now if we solve the system of equations (2.8) by Cramer's rule for the unknowns $c_{in}^{(k)} = x_i$, we obtain

$$| c_{in}^{(k)} | = \left| \frac{\det S_{i+1}}{\det S} \right| = \frac{O(n^{-k'+2i})}{| b_0 n^{-k'} + o(n^{-k'})|} = O(n^{2i}),$$

$i = 0, 1,..., k$. This establishes the lemma.

The $2k$-zero operators defined by the trigonometric kernels $\hat{K}_N^{(k)}$ constructed in Lemma 1 yield approximating functions $\hat{\mathscr{T}}_N^{(k)}(f; x)$ which are themselves trigonometric polynomials in x of degree $N = 2(n-1)(k+1) + k$. The representation of these approximations as trigonometric polynomials may be obtained as follows.

Since J_{nk} is an even trigonometric polynomial of degree $n' = 2(n-1)(k+1)$, it admits the representation

$$J_{nk}(t) = n^{-4k-3} \sum_{i=0}^{n'} \rho_{in}^{(k)} \cos it, \qquad k \geqslant 0,$$

where $\rho_{in}^{(k)} = O(n^{4k+3})$, $i = 0, 1,..., n'$. Therefore,

$$\begin{aligned} \hat{K}_N^{(k)}(t) &= \left[\sum_{i=0}^{k} c_{in}^{(k)} \left(\sin \frac{t}{2}\right)^{2i} \right] J_{nk}(t) \\ &= n^{-4k-3} \left[c_{0n}^{(k)} + \tfrac{1}{2} \sum_{i=1}^{k} c_{in}^{(k)} (1 - \cos t)^i \right] \left[\sum_{i=0}^{n'} \rho_{in}^{(k)} \cos it \right] \\ &= n^{-4k-3} \sum_{i=0}^{N} \sigma_{in}^{(k)} \cos it \end{aligned} \tag{2.10}$$

where the $\{\sigma_{in}^{(k)}\}_{i=0}^{N}$ may be calculated from the coefficients $\{c_{in}^{(k)}\}_{i=0}^{k}$ and the values $\{\rho_{in}^{(k)}\}_{i=0}^{n'}$. Since $J_{nk}(t) = n^{-k}[J_{n0}(t)]^{k+1}$, therefore, for fixed n, the $\rho_{in}^{(k)}$ may be calculated recursively with respect to k, having (Schurer [8])

$$\rho_{in}^{(0)} = \frac{1}{3} \begin{cases} 3i^2 - 6i^2 n - 3i + 4n^3 + 2n, & 1 \leqslant i \leqslant n, \\ i^3 + 6i^2 n - i(12n^2 - 1) + 8n^3 - 2n, & n \leqslant i \leqslant 2(M-1) \end{cases}$$

$$\rho_{on}^{(0)} = \tfrac{1}{3}(2n^3 + n).$$

We then obtain

$$\hat{\mathscr{T}}_N^{(k)}(f; x) = n^{-4k-3} \int_{-\pi}^{\pi} f(t) \left\{ \sum_{i=0}^{N} \sigma_{in}^{(k)} \cos i(t-x) \right\} dt$$

$$= n^{-4k-3} \left\{ a_{on}^{(k)} + \sum_{i=1}^{N} (a_{in}^{(k)} \cos ix + b_{in}^{(k)} \sin ix) \right\}$$

where

$$a_{in}^{(k)} = \sigma_{in}^{(k)} \int_{-\pi}^{\pi} f(t) \cos it, \qquad i = 0, 1, \dots, N,$$

$$b_{in}^{(k)} = \sigma_{in}^{(k)} \int_{-\pi}^{\pi} f(t) \sin it, \qquad i = 1, \dots, N.$$

For example, if $k = 1$, we have from (2.10) the 2-zero kernels

$$\hat{K}_N^{(1)}(t) = n^{-7} \left\{ (c_{on}^{(1)} + \tfrac{1}{2} c_{1n}^{(1)}) \sum_{i=0}^{N-1} \rho_{in}^{(1)} \cos it \right.$$

$$\left. - \tfrac{1}{4} c_n^{(1)} \sum_{i=0}^{N-1} \rho_{in}^{(1)} (\cos(i+1)\, t + \cos(i-1)\, t) \right\}$$

$$= n^{-7} \sum_{i=0}^{N} \sigma_{in}^{(1)} \cos it, \qquad N = 4n - 3.$$

Therefore,

$$\sigma_{in}^{(1)} = (c_{on}^{(1)} + \tfrac{1}{2} c_{1n}^{(1)})\, \rho_{in}^{(1)} - \tfrac{1}{4} c_{1n}^{(1)} \times \begin{cases} \rho_{1n}^{(1)}, & i = 0, \\ 2\rho_{on}^{(1)} + \rho_{2n}^{(1)}, & i = 1, \\ \rho_{i+1,n}^{(1)} + \rho_{i-1,n}^{(1)}, & i = 2, \dots, N-1, \end{cases}$$

and

$$\sigma_{4n-3,n}^{(1)} = -\tfrac{1}{4} c_{1n}^{(1)} \rho_{4n-4,n}^{(1)} .$$

3. Asymptotic Behavior of the $2k$-Zero Operators $\mathscr{T}_n^{(k)}$ and $\mathscr{H}_n^{(k)}$

Critical Degree of Convergence

In this section we show that for the operators $\mathscr{H}_n^{(k)}$ and $\mathscr{T}_n^{(k)}$ defined in Definition 3, Section 1, better degrees of approximation are possible for certain classes of functions. On the other hand, we show that there is a limit, in a certain sense, to this improved degree of approximation.

To state these ideas more precisely, we let $\|\cdot\|$ denote the Tchebycheff norm and make the following definitions.

DEFINITION 4. Let $\{\mathscr{L}_n\}$ be a sequence of operators defined on C_a, and let $\psi(n)$ be a positive function defined on the positive integers. Then $\{\mathscr{L}_n\}$ is said to have *critical degree of convergence* $\psi(n)$ if:

(i) $\psi(n) = o(1)$;

(ii) there exists an $f \in C^\infty$ such that

$$\varliminf_{n\to\infty} \frac{\| \mathscr{L}_n(f; x) - f(x)\|}{\psi(n)} > 0; \qquad \text{and}$$

(iii) for every $f \in C^\infty$, $\| \mathscr{L}_n(f; x) - f(x)\| = O(\psi(n))$.

DEFINITION 5. Let $\psi(n)$ be a critical degree of convergence for a sequence of operators $\{\mathscr{L}_n\}$. If there exists a set of functions $S^* \subseteq C_a$ such that $\| \mathscr{L}_n(f; x) - f(x)\| = O(\psi(n))$ if and only if $f \in S^*$, then S^* is called a *domain of critical degree* for $\{\mathscr{L}_n\}$.

The moduli of continuity of order one and two of an $f \in C_a$ are defined, respectively, as

$$\omega_1(f; h) = \max_{x,t} [| f(x + t) - f(x)|, | t | \leqslant h],$$

$$\omega_2(f; h) = \max_{x,t} [| f(x + t) + f(x - t) - 2f(x)|, | t | \leqslant h],$$

and we note that for any $r \geqslant 0$, $\omega_1(f, rh) \leqslant (1 + r)\, \omega_1(f, h)$. We then define, for every integer $m \geqslant 0$, the following classes of functions:

$$C^m = \{f \in C_a : f \text{ has a continuous } m\text{th derivative}\},$$
$$Y^m = \{f \in C_a : \omega_1(f^{(m)}, h) \leqslant Bh \text{ for some } B > 0\},$$
$$Z^m = \{f \in C_a : f^{(m)} \text{ continuous and } \omega_2(f^{(m)}, h) \leqslant Bh \text{ for some } B > 0\},$$

and note that $C^{m+1} \subset Y^m \subset Z^m \subset C^m$.

To illustrate the concepts of critical degree of convergence and domain of critical degree, consider positive linear operators. Let $\{\mathscr{T}_n\}$ be any sequence of positive trigonometric polynomial operators of degree n (which do not necessarily have to be of the convolution form given in Definition 2). Korovkin [5] has shown that there exists an $f \in C^\infty$ (e.g., one of the functions 1, $\cos x$, $\sin x$) such that $\| \mathscr{T}_n(f; x) - f(x)\| = o(n^{-2})$ is false. It is known, however, that for the positive trigonometric polynomial Jackson operator of degree $2(n - 1)$, defined by

$$\mathscr{T}_n(f; x) = \frac{3}{2\pi n(2n^2 + 1)} \int_{-\pi}^{\pi} f(x + t) \left(\frac{\sin \dfrac{nt}{2}}{\sin \dfrac{t}{2}} \right)^4 dt, \tag{3.1}$$

we have $\| \mathscr{T}_n(f; x) - f(x) \| = O(n^{-2})$ for every $f \in C^2$. Hence $\psi(n) = n^{-2}$ is a critical degree of convergence for the Jackson operators $\{\mathscr{T}_n\}$. Moreover, Lorentz [6] has shown that a domain of critical degree for these operators is Y^1. Also, for the positive operators of the type $\mathscr{H}_n^{(0)}$ defined in Definition 3b, Butzer [1] has shown that $\{\mathscr{H}^{(0)}\}$ has a critical degree of convergence $\psi(n) = n^{-2}$.

The Operators $\mathscr{H}_n^{(k)}$

THEOREM 2. *Let $\{\mathscr{H}_n^{(k)}\}$ be a sequence of the $2k$-zero operators defined in Definition* 3b. *Then $\{\mathscr{H}_n^{(k)}\}$ has a critical degree of convergence $\psi(n) = n^{-2k-2}$. Moreover, if S^* is a domain of critical degree for $\{\mathscr{H}_n^{(k)}\}$, then $Y^{2k+1} \subseteq S^*$.*

Proof. Consider first the local asymptotic behavior of $\mathscr{H}_n^{(k)}(f; x) - f(x)$ for $f \in C^{2k+2}$. Following the method of proof of Butzer [1] we expand $f(t)$ in its Taylor's series with a remainder:

$$f(t + x) - f(x) = \sum_{i=1}^{m} \frac{f^{(i)}(x)}{i!} t^i + \frac{1}{m!} [f^{(m)}(\xi_t) - f^{(m)}(x)] t^m \tag{3.2}$$

where $m = 2k + 2$, ξ_t lies between x and $x + t$, and we assume $f^{(m)}(x) \neq 0$. Let $\theta(t) = f^{(m)}(\xi_t) - f^{(m)}(x)$. Then

$$\begin{aligned} \mathscr{H}_n^{(k)}(f; x) - f(x) &= n \int_{-\infty}^{\infty} [f(t + x) - f(x)] H^{(k)}(nt)\, dt \\ &= n \sum_{i=1}^{m} \frac{f^{(i)}(x)}{i!} \int_{-\infty}^{\infty} t^i H^{(k)}(nt)\, dt \\ &\quad + \frac{n}{m!} \int_{-\infty}^{\infty} \theta(t)\, t^m H^{(k)}(nt)\, dt. \end{aligned} \tag{3.3}$$

Since $H(u)$ is an even function, the summands corresponding to $i = 1, 3, \ldots, 2k - 1$ vanish. From the definition of a $2k$-zero kernel (condition iii of Definition 1) it follows, however, that the summands corresponding to $i = 2, 4, \ldots, 2k$ also vanish. Hence, we may rewrite (3.3):

$$\mathscr{H}_n^{(k)}(f; x) - f(x) = \frac{1}{m!} \left\{ f^{(m)}(x) \int_{-\infty}^{\infty} nt^m H^{(k)}(nt)\, dt + \int_{-\infty}^{\infty} n\theta(t)\, t^m H^{(k)}(nt)\, dt \right\} \tag{3.4}$$

Let I_1 and I_2 denote the first and second integrals in (3.4) respectively. Then

$$I_1 = n^{-m} \int_{-\infty}^{\infty} u^m H^{(k)}(u)\, du = n^{-m} M_{k+1}^{H}(H^{(k)}). \tag{3.5}$$

Now since $f^{(m)}(t)$ is continuous and bounded, for every $\epsilon > 0$ there exists a δ_ϵ such that $|\theta(t)| < \epsilon$ when $0 \leqslant |t| < \delta_\epsilon$; and there is a $B > 0$ such that $|\theta(t)| \leqslant B$ for all t. Given $\epsilon > 0$, write

$$I_2 = \left(\int_{|t|\leqslant\delta_\epsilon} + \int_{|t|>\delta_\epsilon}\right) n\theta(t)\, t^m H^{(k)}(nt)\, dt.$$

Let I_{21} and I_{22} denote the first and second of these integrals, respectively. Then

$$\begin{aligned}|I_{21}| &\leqslant \epsilon n \int_{|t|\leqslant\delta_\epsilon} t^m\, |H^{(k)}(nt)|\, dt = \epsilon n^{-m} \int_{|u|\leqslant\delta_\epsilon} u^m\, |H^{(k)}(u)|\, du \\ &= \epsilon n^{-m} M_{k+1}^H(|H^{(k)}|)\end{aligned}$$

and

$$|I_{22}| \leqslant Bn \int_{|t|\geqslant\delta_\epsilon} t^m\, |H^{(k)}(nt)|\, dt = Bn^{-m} \int_{|u|\geqslant n\delta_\epsilon} u^m\, |H^{(k)}(u)|\, du.$$

From Definition 3(b) we have that $M_{k+1}(|H^{(k)}|)$ exists. Therefore, there is an $N_\epsilon > 0$ such that for $n > N_\epsilon$ we have

$$\int_{|u|\geqslant n\delta_\epsilon} u^m\, |H^{(k)}(u)|\, du < \epsilon.$$

Hence, $|I_{22}| \leqslant \epsilon Bn^{-m}$ and

$$|I_2| \leqslant |I_{21}| + |I_{22}| \leqslant \epsilon B' n^{-m}$$

where $B' = B + M_{k+1}^H(|H^{(k)}|)$. From this result, (3.4) and (3.5), we have, for each fixed x such that $f^{(m)}(x) \neq 0$, that

$$\lim_{n\to\infty} n^m[\mathscr{H}_n^{(k)}(f; x) - f(x)] = \frac{f^{(m)}(x)}{m!} M_{k+1}^H(H^{(k)}). \tag{3.6}$$

Thus, if $f^{(m)}(t) \not\equiv 0$, then (3.6) implies that

$$\| \mathscr{H}_n^{(k)}(f; x) - f(x) \| \neq o(n^{-m}), \qquad m = 2k + 2. \tag{3.7}$$

Note that if we did have $\| \mathscr{H}_n^{(k)}(f; x) - f(x)\| = o(n^{-m})$, then $f^{(m)}(t) \equiv 0$, in which case $f(t)$ must be a polynomial of degree $m - 1 = 2k + 1$.

Now consider the global asymptotic behavior of the difference $\mathscr{H}_n^{(k)}(f; x) - f(x)$; we let $f \in Y^{2k+1}$. By an argument similar to the one used in deriving (3.4) we have

$$\mathscr{H}_n^{(k)}(f; x) - f(x) = \frac{n}{(m-1)!} \int_{-\infty}^{\infty} \theta(t)\, t^{m-1} H^{(k)}(nt)\, dt. \tag{3.8}$$

Now

$$
\begin{aligned}
|\theta(t)| = |f^{(m-1)}(\xi_t) - f^{(m-1)}(x)| &\leqslant \omega_1(f^{(m-1)}, |\xi_t - x|) \\
&\leqslant (1 + n|\xi_t - x|)\, \omega_1\left(f^{(m-1)}, \frac{1}{n}\right) \\
&\leqslant (1 + n|t|)\, \omega_1\left(f^{(m-1)}, \frac{1}{n}\right).
\end{aligned}
$$

Therefore, using (3.8), we obtain

$$
\begin{aligned}
|\mathscr{H}_n^{(k)}(f; x) - f(x)| &\leqslant \frac{n}{(m-1)!}\, \omega_1\left(f^{(m-1)}, \frac{1}{n}\right) \int_{-\infty}^{\infty} (|t^{m-1}| + n|t^m|)\, |H^{(k)}(nt)|\, dt \\
&= \frac{n^{-m+1}}{(m-1)!}\, \omega_1\left(f^{(m-1)}, \frac{1}{n}\right) \int_{-\infty}^{\infty} (|u^{m-1}| + |u^m|)\, |H^{(k)}(u)|\, du \\
&= O\left(n^{-2k-1} \omega_1\left(f^{(m-1)}, \frac{1}{n}\right)\right).
\end{aligned}
$$

But since $f \in Y^{2k+1}$, $\omega_1(f^{(2k+1)}, 1/n) = O(n^{-1})$. Hence,

$$
\| \mathscr{H}_n^{(k)}(f; x) - f(x) \| = O(n^{-2k-2}). \tag{3.9}
$$

Relation (3.9) together with (3.7) imply that $\psi(n) = n^{-2k-2}$ is a critical degree of convergence for $\{\mathscr{H}_n^{(k)}\}$. Moreover, (3.9) implies that $Y^{2k+1} \subseteq S^*$, where S^* is a domain of critical degree for $\{\mathscr{H}_n^{(k)}\}$. This concludes the proof of the theorem.

The Trigonometric Operators $\mathscr{T}_n^{(k)}$

Recall that the assertion of Theorem 2 concerning critical degree of convergence holds for *any* sequence of $2k$-zero operators of the type $\mathscr{H}_n^{(k)}$. This is not too surprising since the $\mathscr{H}_n^{(k)}$ are a very special type of $2k$-zero operators whose kernels are generated from a single kernel function. For sequences of $2k$-zero trigonometric polynomial operators, however, it is not necessarily true that a degree of convergence as good as n^{-2k-2} can be achieved for sufficiently smooth functions, even if the degree of convergence improves as k increases (as for example, is the case for the sequences of operators defined by the 2-zero and 4-zero kernels $V_n^{(1)}$ and $V_n^{(2)}$ constructed in Section 2). On the other hand, it is true that every sequence $\{\mathscr{T}_n^{(k)}\}$ satisfies condition (ii) of Definition 4 with $\psi(n) = n^{-2k-2}$, i.e., there exists an $f \in C_\pi$ such that $\underline{\lim}_{n\to\infty}\, n^{2k+2} \| \mathscr{T}_n^{(k)}(f; x) - f(x) \| > 0$. Furthermore, a particular sequence of $2k$-zero trigonometric polynomial operators can be constructed that does indeed have a critical degree of convergence n^{-2k-2}. These assertions are contained in the following lemma and theorems.

LEMMA 2. *Let* $\{\mathscr{T}_n^{(k)}\}$ *be a sequence of* $2k$*-zero trigonometric polynomial operators* $k \geqslant 1$, *as defined in Definition* 3a. *Then for* $g(t) = (\sin(t/2))^{2k+2}$, *we have* $\| \mathscr{T}_n^{(k)}(g; x) - g(x)\| \geqslant cn^{-2k-2}$, $c > 0$.

A proof of this lemma is given in Hoff [3]. The proof proceeds by induction on k and depends upon showing that all the zeros of the kernels satisfy $|\alpha_{ni}| \geqslant c_1 n^{-1}$, $c_1 > 0$, $i = 1,..., k$, and then employing the well known result that if $T_{n+1}(x)$ is the $(n+1)$st degree trigonometric polynomial of best approximation to $|\sin x|$ on $[-\pi, \pi]$, then $\| T_{n+1}(x) - |\sin x| \| \geqslant c_2 n^{-1}$, $c_2 > 0$ (Korovkin [5]).

From this lemma we immediately obtain

THEOREM 3a. *Let* $\{\mathscr{T}_n^{(k)}\}$ *be a sequence of* $2k$*-zero trigonometric polynomial operators*, $k \geqslant 1$, *as defined in Definition* 3a. *Then there is an* $f \in C^\infty$ *such that*

$$\varlimsup_{n\to\infty} n^{2k+2} \| \mathscr{T}_n^{(k)}(f; x) - f(x)\| > 0.$$

Remark. The paper by Butzer, Nessel and Scherer [2] establishes this theorem using a more general approach. It is shown that if $\{K_n(u)\}$ is a general sequence of even, trigonometric polynomials of degree n such that $K_n(0) > 0$ for n sufficiently large, and if $K_n(u)$ has exactly $2k$ changes of sign for each $n > 2k$, then at least one of the $k+2$ sequences

$$\{n^{2k+2} \| \mathscr{L}_n(\cos iu; x) - \cos ix \|\}, \qquad 0 \leqslant i \leqslant k+1,$$

does not tend to zero as $n \to \infty$, where

$$\mathscr{L}_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, K_n(x-t)\, dt$$

and $\|\cdot\|$ can be taken as the Tchebycheff or as any L_p norm, $1 \leqslant p < \infty$. Similar types of results are also given for nonsymmetric $K_n(u)$ and for K_n such $K_n(0) < 0$ for sufficiently large n.

We now prove an analogous theorem to Theorem 2 for a particular sequence of $2k$-zero trigonometric polynomial operators. The following facts are needed (Hoff [3]):

Fact 4. Let k, m, and i be positive integers such that $k \geqslant 2$ and $m + i < 2k$. Then

$$n^{-2k+1} \int_{-\pi}^{\pi} |t|^m \left|\sin \frac{t}{2}\right|^i \left(\frac{\sin \dfrac{nt}{2}}{\sin \dfrac{t}{2}} \right)^{2k} dt = O(n^{-m-i}).$$

Fact 5. Let $h_i(t) = (\sin(t/2))^i$, $i = 1,\dots, 2k + 1$. Then

$$t^i = \left(2 \sin \frac{t}{2}\right)^i - \sum_{j=i+1}^{2k+1} a_{ij} t^j - 2^i \varphi_i(t)\, t^{2k+2}$$

where

$$a_{ij} = \frac{2^i}{j!} h_i^{(j)}(0) \qquad \text{and} \qquad \varphi_i(t) = \frac{1}{(2k+2)!} h_i^{(2k+2)}(\xi_{it}), \tag{3.10}$$

and ξ_{it} is between 0 and t.

We also make use of a well known result due to Zygmund [10]:

Fact 6. Let $T_n(x)$ be the nth degree trigonometric polynomial of best Tchebycheff approximation to the periodic function $f(x)$. Then for any integer $m \geqslant 1$, $\| T_n(x) - f(x)\| = O(n^{-m-1})$ if and only if $f \in Z^m$.

THEOREM 3b. *There exists a sequence $\{\hat{\mathscr{T}}_n^{(k)}\}$ of 2k-zero trigonometric polynomial operators which has a critical degree of convergence $\psi(n) = n^{-2k-2}$, $k \geqslant 1$, and a domain of critical degree, S^*, satisfying $Y^{2k+1} \subseteq S^* \subseteq Z^{2k+1}$.*

Proof. Let $\{\hat{\mathscr{T}}_N^{(k)}\}$ be the sequence of $2k$-zero trigonometric polynomial operators defined by the kernels $\hat{K}_N^{(k)}$ constructed in Lemma 1:

$$\hat{K}_N^{(k)}(t) = \left[\sum_{i=0}^{k} c_{in}^{(k)} \left(\sin \frac{t}{2}\right)^{2i} \right] J_{nk}(t),$$

where $N = 2(n-1)(k+1) + k$ and $J_{nk}(t)$ is defined by (2.3). We show that $\| \hat{\mathscr{T}}_N^{(k)}(f; x) - f(x)\| = O(n^{-2k-2})$ for every $f \in C^\infty$. We in fact show that this holds for every $f \in Y^{2k+1}$.

Let $f \in Y^{2k+1}$; expand $f(t)$ in its Taylor series with remainder at $x \in [-\pi, \pi]$:

$$f(t + x) - f(x) = \sum_{i=1}^{m} \frac{f^{(i)}(x)}{i!} t^i + \frac{1}{m!} \theta(t)\, t^m \tag{3.11}$$

where $m = 2k + 1$, $\theta(t) = f^{(m)}(\xi_t) - f^{(m)}(x)$, and ξ_t lies between x and $t + x$. From the representation of t^i, $i = 1,\dots, 2k$, given in Fact 5, (3.11) may be rewritten as

$$f(t + x) - f(x) = \sum_{i=1}^{m} b_i \left(\sin \frac{t}{2}\right)^i + \frac{1}{m!} \theta(t)\, t^m + \left(\sum_{i=1}^{m} b_i \varphi_i(t)\right) t^{m+1}$$

where $\varphi_i(t)$ is defined in (3.10) and each coefficient b_i is a sum of terms involving the derivatives of f at x and the coefficients a_{ij} defined in (3.10).

Therefore, since $M_0{}^T(\hat{K}_N^{(k)}) = 1$, we have

$$\hat{\mathscr{T}}_N^{(k)}(f; x) - f(x) = \int_{-\pi}^{\pi} [f(x+t) - f(x)]\, \hat{K}_N^{(k)}(t)\, dt$$

$$= \sum_{i=1}^{m} b_i \int_{-\pi}^{\pi} \left(\sin\frac{t}{2}\right)^i \hat{K}_N^{(k)}(t)\, dt + \frac{1}{m!}\int_{-\pi}^{\pi} \theta(t)\, t^m \hat{K}_N^{(k)}(t)\, dt$$

$$+ \sum_{i=1}^{m} b_i \int_{-\pi}^{\pi} \varphi_i(t)\, t^{m+1} \hat{K}_N^{(k)}(t)\, dt. \tag{3.12}$$

The first sum in (3.12) vanishes. This follows from the fact that the terms with i even vanish since $M_j{}^T(\hat{K}_N^{(k)}) = 0, j = 1,..., k$, and the terms with i odd vanish since $(\sin(t/2))^i\, \hat{K}_N^{(k)}(t)$ is an odd function for $i = 1, 3,..., m$.

As to the second term in (3.12), since

$$|\,\theta(t)| \leqslant \omega_1(f^{(m)}, |\,\xi_t - x\,|) \leqslant (1 + n\,|\,t\,|)\, \omega_1\left(f^{(m)}, \frac{1}{n}\right),$$

we have

$$\left|\int_{-\pi}^{\pi} \theta(t)\, t^m \hat{K}_N^{(k)}(t)\right| \leqslant \omega_1\left(f^{(m)}, \frac{1}{n}\right)\int_{-\pi}^{\pi} |(1 + n\,|\,t\,|)\, t^m \hat{K}_N^{(k)}(t)|\, dt$$

$$\leqslant \omega_1\left(f^{(m)}, \frac{1}{n}\right) \sum_{i=0}^{k} |\,c_{in}^{(k)}\,| \int_{-\pi}^{\pi} (|\,t\,|^m + n t^{m+1}) \left(\sin\frac{t}{2}\right)^{2i} J_{nk}(t)\, dt.$$

But, by Lemma 1, $|\,c_{in}^{(k)}\,| = O(n^{2i})$, $i = 0,..., k$, and, by Fact 4,

$$\int_{-\pi}^{\pi} |\,t\,|^j \left(\sin\frac{t}{2}\right)^{2i} J_{nk}(t)\, dt = O(n^{-j-2i}), \qquad j + 2i < 4k + 4.$$

Hence,

$$\left|\int_{-\pi}^{\pi} \theta(t)\, t^m \hat{K}_N^{(k)}(t)\right| = O\left(\omega_1\left(f^{(m)}, \frac{1}{n}\right) n^{-m}\right).$$

Finally, the last sum in (3.12) is $O(n^{-m-1})$. This follows since for $i = 1,..., m$, $|\,\varphi_i(t)| \leqslant B$, where B is some constant depending on k, and hence, by Lemma 1 and Fact 4,

$$\left|\int_{-\pi}^{\pi} \varphi_i(t)\, t^{m+1} \hat{K}_N^{(k)}(t)\, dt\right| \leqslant B \sum_{j=0}^{k} |\,c_{jn}^{(k)}\,| \int_{-\pi}^{\pi} t^{m+1} \left(\sin\frac{t}{2}\right)^{2j} J_{nk}(t)\, dt$$

$$= O(n^{-m-1}), \qquad i = 1, 2,..., m.$$

Thus, from these results and (3.12), we have $\| \hat{\mathscr{T}}_N^{(k)}(f; x) - f(x) \| = O(\omega_1(f^{(m)}, 1/n)\, n^{-m}) + O(n^{-m-1})$. But since $f \in Y^m$, $\omega_1(f^{(m)}, 1/n) = O(1/n)$, and so $\| \hat{\mathscr{T}}_N^{(k)}(f; x) - f(x) \| = O(n^{-m-1})$, $N = 2(n-1)(k+1) + k$, $m = 2k+1$. On the other hand, by Theorem 3a, there exists an $f \in C^\infty$ such that $\lim_{n\to\infty} n^{2k+2} \| \hat{\mathscr{T}}_N^{(k)}(f; x) - f(x) \| > 0$. Hence, $\{\hat{\mathscr{T}}_N^{(k)}\}$ has a critical degree of convergence $\psi(n) = n^{-2k-2}$.

Since $f \in Y^{2k+1}$ implies $\| \hat{\mathscr{T}}_N^{(k)}(f; x) - f(x) \| = O(n^{-2k-2})$, we have $Y^{2k+1} \subseteq S^*$, where S^* is a domain of critical degree for $\{\hat{\mathscr{T}}_N^{(k)}\}$. But by Fact 6 (Zygmund's theorem), we also have that

$$\| T_N(x) - f(x) \| \leqslant \| \hat{\mathscr{T}}_N^{(k)}(f; x) - f(x) \| = O(n^{-2k-2}) \quad \text{implies} \quad f \in Z^{2k+1},$$

where $T_N(x)$ is the trigonometric polynomial of degree N of best Tchebycheff approximation to $f(x)$. Hence, $S^* \subseteq Z^{2k+1}$. This concludes the proof of the theorem.

Final Remarks

Since $\| T_n(x) - f(x) \| = O(n^{-m-1})$ does not imply $f \in Y^m$, we were unable, in the proof of Theorem 3b, to characterize S^* as being exactly Y^{2k+1}, using the type of argument employed there. But, as was noted previously, a domain of critical degree for the positive Jackson operators (3.1) is Y^1. This leads one to conjecture that a domain of critical degree for the $2k$-zero operators $\{\hat{\mathscr{T}}_N^{(k)}\}$ is indeed Y^{2k+1}. The methods of proof employed for the positive Jackson operators, however, seem intractable for these $2k$-zero operators.

A simple example is given in Tables I and II, which compare the errors in approximation to $f(t) = [\cos(t/2)]^6$ at $t = 0$ using the positive, 2-zero and 4-zero operators defined by the kernels $W_n^{(i)}$ and $V_n^{(i)}$, $i = 0, 1, 2$, constructed in Section 2. By Theorem 2, we can expect the operators defined by $W_n^{(0)}$, $W_n^{(1)}$

TABLE I

$W_n^{(i)}$—Approximation Errors for $(\cos t/2)^6$ at $t = 0$ Using Positive, 2-Zero and 4-Zero Operators ($i = 0, 1, 2$)

n	Positive	2-Zero	4-Zero
1	0.2501660	0.0825120	0.0282779
2	0.0833193	0.0092754	0.0010351
3	0.0394717	0.0020808	0.0001086
4	0.0227269	0.0006894	0.0000206
5	0.0147058	0.0002885	0.0000056
6	0.0102736	0.0001408	0.0000019
7	0.0075757	0.0000765	0.0000007

TABLE II

$V_n^{(i)}$—Approximation Errors for $(\cos t/2)^6$ at $t = 0$ Using Positive, 2-Zero and 4-Zero Operators ($i = 0, 1, 2$)

n	Positive	2-Zero	4-Zero
1	0.453125	0.124999	0.015624
2	0.343749	0.081249	0.008928
3	0.278124	0.057142	0.005795
4	0.233928	0.042410	0.003719
5	0.202008	0.032737	0.002603
6	0.177827	0.026040	0.001893
7	0.158853	0.021211	0.001400

and $W_n^{(2)}$ to achieve a degree of convergence n^{-2}, n^{-4} and n^{-6}, respectively, for sufficiently smooth functions. Table I shows that such degrees of convergence were indeed obtained. Theorem 3b, however, does not assure us of achieving degrees of convergence as good as those for the operators defined by the trigonometric polynomial kernels $V_n^{(i)}$. It can be shown, in fact, that the operators defined by $V_n^{(0)}$, $V_n^{(1)}$ and $V_n^{(2)}$ give a degree of convergence not better than n^{-1}, n^{-2} and n^{-3}, respectively. Table II shows that these degrees of convergence were actually obtained. A straightforward numerical integration procedure was used to compute the approximations.

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